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A new expression for the D-dimensional Fierz coefficients

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Abstract. It is shown that (up to normalisation) the coefficients of a D-dimensional Fierz transformation are characters of the symmetric group. Some connections with a geometric setting for these coefficients are made.

1. Introduction

The decomposition of the tensor product of two irreducible representations of a group into irreducible representations may be done in many ways. The coefficients that appear in such an expansion are the Clebsch–Gordan or 3-j Wigner symbols and similar decompositions can be carried out for arbitrary tensor products. Of particular importance in physics are the tensor products involving the spin groups. Here the decomposition of the fourfold product is known as a Fierz transformation; the coefficients appearing are the Fierz or 6-j coefficients. This decomposition provides identities amongst biquadratic Lorentz covariant combinations of spinors which occur in many contexts: Fierz (1937) employed them in a four-fermion treatment of β decay; today they are widely used in supersymmetric calculations. The purpose of this note is to characterise the Fierz coefficients appearing in this decomposition as characters of the symmetric group and comment upon this connection.

Early studies of Fierz transformations were often done by actual calculation with a specific representation of the gamma matrices. Harish-Chandra (1945), for example, started with Eddington's (1936) defining relations and derived to the Fierz identities by quite calculational means. Other methods were however available, particularly after the incisive work of Brauer and Weyl (1935) in examining spinors in D dimensions. Pauli (1936) derived in an algebraic and representation independent manner many properties of spinors and Fierz transformations; much of his notation is still used today. Case (1955) made a detailed study of Fierz transformations by essentially algebraic means. He characterised the coefficients of the transformation matrix as coefficients of a certain polynomial. To be more precise we must specify our notation.

The 'elementary' Fierz transformations are the scalar identities

$$(\bar{\psi}_1\Gamma^{(I)}\psi_2)(\bar{\phi}_1\Gamma_{(I)}\phi_2) = -(1/2^{d/2})\sum_K a^D_{IK}(\bar{\psi}_1\Gamma^{(K)}\phi_2)(\bar{\phi}_1\Gamma_{(K)}\psi_2). \tag{1}$$

If D is the space–time dimension then $d \equiv 2[D/2]$ is the largest even integer less than or equal to D; $2^{d/2}$ is the dimension of the spinor representation. By $\Gamma^{(I)}$ we denote the Lorentz covariant sum of I products of gamma matrices, that is a sum over the $\binom{d}{I}$ possible combinations

$$\Gamma^{I,\mu_1\mu_2\ldots\mu_I} \equiv \Gamma^{\mu_1}\ldots.\Gamma^{\mu_I}, \qquad \mu_1<\mu_2<\mu_I, \qquad \Gamma_0 \equiv 1. \tag{2}$$

The order taken in (2) is a matter of choice; such a labelling gives us a basis for the $2^{d/2}\times 2^{d/2}$ matrices. The sum in (1) is over the classes of independent $\Gamma^{(K)}$: for even D it involves $D+1$ terms; for odd D it involves $d/2+1$ terms. By lowering matrices with the metric tensor appropriate to the space–time under consideration (1) is independent of signature. Lastly the overall minus sign appearing in (1) comes from taking spinors to anticommute; its appearance varies between authors.

The coefficients a^D_{IK} are the (normalised) Fierz coefficients. The work of Case shows these to be given by

$$(-1)^{IK}a^D_{IK}=\text{coeff}_{x^I}\text{ in }(1+x)^{D-K}(1-x)^K=\sum_j(-1)^j\binom{D-K}{I-j}\binom{K}{j}. \tag{3}$$

(Actually this is a more symmetric form of Case's work where the differences between even and odd space–time dimension D are accentuated by his choice of basis $\Gamma^{(I)}$.) Indeed this coefficient may be expressed in terms of a Jacobi polynomial evaluated at a given argument. With

$$P_n^{(\alpha,\beta)}(x)=2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k, \tag{4a}$$

$$(-1)^{IK}a^D_{IK}=2^IP_I^{(D-I-K,K-I)}(0). \tag{4b}$$

This gives the connection with Kennedy's (1982) generalisation of spinor identities to non-integral dimension to be used in the framework of dimensional regularisation. The appearance of Jacobi polynomials here is not fortuitous: $P_l^{(m,n)}(z)$ appears generally when considering the matrix elements of the irreducible unitary (infinite-dimensional) operator representations of SL(2, $\mathbb{C}$); for $\alpha=\beta=\frac{1}{2}$ they also have an interpretation as characters of SU(2) obviously connected with the ultra-spherical polynomials (Askey 1975); their discrete difference analogue also appears in the study of Clebsch–Gordan coefficients (Vilenkin 1974). A similar connection with SL(2, $\mathbb{C}$) and Fierz coefficients was made by Corrigan *et al* (1973).

We show here that the coefficients a^D_{IK} have a further group theoretic interpretation. Namely we establish and comment upon the following lemma.

Lemma. For all D the Fierz coefficients a^D_{IK} are given in terms of the characters χ of the symmetric group on $D+1$ symbols:

$$(-1)^{IK}a^D_{IK}=\chi^{[D+1-I,1^I]}_{1^{D+1+2K}2^K}. \tag{5}$$

The notation employed in (5) is as follows: the superscript in square brackets of the character function χ on the right-hand side of this equation denotes the partition of $D+1$ symbols we are interested in. Here the Young tableaux corresponding to this partition would have $D+1-I$ boxes in a row with I single boxes placed beneath the first. The subscript of χ gives the conjugacy class of the group element whose character we are evaluating. The conjugacy classes of the symmetric group are determined by their cycle type and here we have $D+1-2K$ one-cycles and K two-cycles. Partitions of the symmetric group are frequently used to classify the representations of the orthogonal and unitary groups (Boerner 1963), so perhaps this expression of the Fierz or 6-j coefficients in terms of characters of the symmetric group should not be surprising.

We will now prove this lemma in two ways. In § 2 it will be shown directly starting with Case's characterisation. Some immediate consequences and symmetries of the Fierz coefficients will then be noted. In § 3 the lemma will be shown by a more circuitous but nonetheless interesting route. There has been one further line of investigation of Fierz transformations which we have not mentioned as yet. Associated with a spin representation there is a natural finite group present, the multiplicative group generated by the gamma matrices (Boerner 1963, Braden 1984). de Vries and Van Zanten (1970) showed how the Fierz coefficients may be expressed in terms of invariants of this group. Starting from this perspective the lemma will be demonstrated and further insights gained. This discussion will be done in § 4 after the connection between the characters of finite groups and the Fierz coefficients is shown in § 3.

2. A direct proof

The characters of the symmetric group can be evaluated in many ways (James 1978). The generating function of Frobenius gives the values of all characters for a particular class, while the Schur function is associated with the particular character and displays its values for all classes. Considering the matrix of entries $(a^D)_{IK}$ the former may be viewed as establishing the columns and the latter the rows. Both lead, using the recurrents of the appropriate symmetric functions, to the lemma. Here we shall apply the Murnagham–Nakayama formula. Most directly, remove a two hook from the initial partition:

$$\chi^{[D+1-I,1^I]}_{1^{D+1-2K}2^K} = \chi^{[D-1-I,1^I]}_{1^{D+1-2K}2^{K-1}} - \chi^{[D+1-I,1^{I-2}]}_{1^{D+1-2K}2^{K-1}}. \tag{6}$$

Continuing in this manner we get

$$\chi^{[l-r,1^r]}_{1^{\alpha},2^{\alpha_2}} = \sum_{j=0} (-1)^j \binom{\alpha_2}{j} \chi^{[l-r-2(\alpha_2-j),1^{r-2j}]}_{1^{\alpha_1}} \tag{7}$$

$$l=\alpha_1+2\alpha_2, \qquad r-2j \geqslant 0, \qquad l-r-2(\alpha_2-j)-1 \geqslant 0,$$

where the restrictions on the sum rise from the requirement that the removal of a two hook must be a partition. Now the character appearing in (7) is just the dimension of the representation. For a hook this is particularly easy to evaluate

$$\chi^{[\alpha_1-(r-2j),1^{r-2j}]}_{1^{\alpha_1}} = \binom{\alpha_1-1}{r-2j} \tag{8}$$

and so

$$\chi^{[l-r,1^r]}_{1^{\alpha_1}2^{\alpha_2}} = \sum_{j=0}^{[r/2]} (-1)^j \binom{\alpha_1-1}{r-2j}\binom{\alpha_2}{j} \tag{9}$$

where we interpret $\binom{a}{b}$ as zero if $b<0$ or $b>a$. Finally we have

$$\sum_{j=0}^{[r/2]} (-1)^j \binom{D-2K}{r-2j}\binom{K}{j} = \sum_{j=0}^{r} (-1)^j \binom{D-K}{r-j}\binom{K}{j}. \tag{10}$$

Combining these results we have for the particular case at hand

$$\chi^{[D+1-I,1^I]}_{1^{D+1-2K}2^K} = \sum_j (-1)^j \binom{D-K}{I-j}\binom{K}{j} = (-1)^{IK} a^D_{IK} \tag{11}$$

where we have used Case's result (3). This establishes the lemma.

The derivation of (11) by deleting a two hook from the initial partition is particularly expedient though other routes are possible. If we had initially deleted a one hook this would have given us the usual branching theorem for characters

$$\chi^{[D+1-I,1^I]}_{1^{D+1-2K}2^K}=\chi^{[D-I,1^I]}_{1^{D-2K}2^K}+\chi^{[D+1-I,1^{I-1}]}_{1^{D-2K}2^K}. \tag{12}$$

Again we can derive (11). Both (6) and (12) give us relations amongst Fierz coefficients:

$$a^D_{IK}=a^{D-1}_{I,K}+(-1)^K a^{D-1}_{I-1,K}, \tag{13a}$$

$$a^D_{IK}=(-1)^I a^{D-2}_{I,K-1}+(-1)^{I+1}a^{D-2}_{I-2,K-1}. \tag{13b}$$

Applying (13a) twice and equating with (13b) gives relations amongst the elements of the Fierz matrix. While such relations are not of great importance, they do enable the rapid computation of a Fierz matrix of any dimension and provide inductive defining relations.

A useful consequence of representing the Fierz coefficients in terms of characters is that the overall symmetries of the Fierz transformation become obvious. We have for conjugate partitions

$$\chi^{[D+1-I,1^I]}=\chi^{[1^{D+1}]}\chi^{[I+1,1^{D-I}]}, \tag{14}$$

$$a^D_{IK}=(-1)^{K[1+D]}a^D_{D-I,K}, \tag{15}$$

as well as (from (11))

$$a^D_{IK}=(-1)^{I[D+1]}a^D_{I,D-K}. \tag{16}$$

These symmetries are of course related to $\Gamma^{(d)}$ and are discussed in § 3. At this stage we direct the reader to table 1 where the Fierz coefficients are given for $D=2$, 3, 4, 5 to see how these symmetries are manifest.

Strictly speaking, Case's result (3) was only for even dimensions. His choice of basis for odd dimensions gives a permutation of the Fierz coefficients here and obscures the underlying symmetry. We conclude this section by showing how the basis chosen here gives (3), and in so doing clarify the differences between even and odd dimensions. In odd dimensions the product $\Gamma^{(d)}$ is, up to a scalar multiple, the representation for the $(2[D/2]+1)$th gamma matrix, i.e. for the 'fifth' axis $\Gamma^5=\Gamma^1\Gamma^2\Gamma^3\Gamma^4$. When we now

Table 1. $a^D_{IK}=(-1)^{IK}\chi^{[D+1-I,1^I]}_{1^{D+1-2K}2^K}$.

$D=2$

I/K	0	1	2
0	1	1	1
1	2	0	−2
2	1	−1	1

$D=3$

I/K	0	1
0	1	1
1	3	−1

$D=4$

I/K	0	1	2	3	4
0	1	1	1	1	1
1	4	−2	0	2	−4
2	6	0	2	0	6
3	4	2	0	−2	−4
4	1	−1	1	−1	1

$D=5$

I/K	0	1	2
0	1	1	1
1	5	−3	1
2	10	2	−2

perforem the Lorentz covariant sum (1), $\Gamma^1\Gamma^2$ and $\Gamma^1\Gamma^5$ are now in the same class. It is this equivalence which leads to a reduction in the size of the Fierz matrix. Further, just by grouping these terms together in the sums, we see that

$$\begin{aligned} a^{d+1}_{I,K} &= a^d_{I,K} + a^d_{d+1-I,K}, \qquad I, K \leqslant [d/2], \\ &= a^d_{I,K} + (-1)^K a^d_{I-1,K} \end{aligned}$$

where we have used (15) and the fact that d is even. But this is just the recursion (13a) and from this (3) results.. We obtain then a symmetric form (3) for the Fierz coefficients in both even and odd dimensions.

3. A group theoretic approach

We shall now establish the lemma by an alternative route. As previously mentioned a finite group may be associated with a spin representation, the multiplicative group generated by the gamma matrices. This group may be though of as the double cover of the (Abelian) group generated by reflections of the coordinate axes (Braden 1982) and its representation theory tells us a great deal about the spin representation. Indeed, it gives us the Racah and Fierz coefficients (de Vries and van Zanten 1970) as well as the reality properties of the spin representations (Braden 1982).

In this section we will establish the connection between this group and the Fierz coefficients.

Let us call the finite group G. The properties we shall need are as follows: G has order 2^{D+1}. If we denote

$$\Delta \equiv \Gamma^{(D)} \equiv \Gamma^1\Gamma^2 \ldots \Gamma^D$$

then the structure of G depends on whether D is even or odd.

(a) If $D=2m$ then G has $2^{2m}+1$ equivalence classes $\{+1\}$, $\{-1\}$, $\{\pm\Gamma^{(I)}\}$ for $1 \leqslant I \leqslant D$. G has 2^{2m} one-dimensional representations and $[G, G] = Z_2 = \langle 1, -1\rangle = Z(G)$. G has one 2^m-dimensional representation.

(b) If $D=2m+1$ then G has $2^{2m+1}+2$ equivalence classes $\{+1\}$, $\{-1\}$, $\{\pm\Gamma^{(I)}\}$, $\{+\Delta\}$, $\{-\Delta\}$, for $1 \leqslant I \leqslant d$. G has 2^{2m+1} one-dimensional representations and $[G, G] = \langle 1, -1\rangle = Z(G) = \langle 1, -1, \Delta, -\Delta\rangle$. G has two 2^m-dimensional representations. Here $Z(G)$ is the centre of G.

It is useful to label the one-dimensional representations. A convenient choice is following: if χ_I is the character of (1_I) then it is given by

$$\chi_I(\Gamma^B)1 = \Gamma^B\Gamma^A(\Gamma^B)^{-1}(\Gamma^A)^{-1}. \tag{17}$$

Being in the commutator subgroup the right-hand side is ± 1. Here Γ^A is some element of our basis of gamma matrices, say $\Gamma^{\mu_1\cdots\mu_3}$. Then the right-hand side of (17) shows that if $\Gamma^{\mu_j} \in \Gamma^B$ it anticommutes with all those $\mu_r \in A$ different from μ_j. Thus

$$\chi_A(\Gamma^B) = \prod_{\mu_j \in B} (-1)^{\#\mu\text{'s in } A \text{ different } \mu_j} = \chi_B(\Gamma^A). \tag{18}$$

That (17) labels distinct representations is seen as follows. Suppose $\chi_A(g) = \chi_{A'}(g)$, $g \in G$. Then $\Gamma_A^{-1}\Gamma_{A'} \in Z(G)$. But for a one-dimensional representation the centre is trivial and so $\Gamma_A = \Gamma_{A'}$.

As an example the character table for the $D=2$ group is given in table 2. (This group is either the quaternion group or dihedral group of order 8.)

Table 2. The characters of the dihedral or quaternion group showing the labelling of representations. Summing the elements in the darkened boxes gives us the entries for $D=2$ in table 1.

Rep \ Class	1	-1	$\pm\Gamma_1$	$\pm\Gamma_2$	$\pm\Gamma_1\Gamma_2$
1	1	1	1	1	1
1_1	1	1	1	−1	−1
1_2	1	1	−1	−1	−1
1_{12}	1	1	−1	−1	1
2	2	−2	0	0	0

Now if we denote

$$|A| = s \qquad \text{when } \Gamma^A = \Gamma^{\mu_1 \cdots \mu_s} \tag{19}$$

then we can derive from (17)

$$\chi_A(\Delta\Gamma^B) = (-1)^{|A|(D+1)}\chi_A(\Gamma^B), \tag{20}$$

$$\chi_{\Delta A}(\Gamma^B) = (-1)^{|B|(D+1)}\chi_A(\Gamma^B), \tag{21}$$

$$\chi_A(\Gamma^B) = (-1)^{|A|\cdot|B|-r}, \tag{22}$$

$$\chi_A(g)\chi_B(g) = \chi_{A^{-1}B}(g). \tag{23}$$

Here r is the number of Γ^μ's common to both. Equations (20) and (21) are the basis for (15) and (16).

The Fierz coefficients are readily expressed in terms of these one-dimensional characters. We have

$$a^D_{IK} = (1/2^{d/2}) \operatorname{Tr}[\Gamma^{(I)}\Gamma^K\Gamma_{(I)}\Gamma^{K^{-1}}] = \sum_{A\in I} \chi_A(\Gamma^K). \tag{24}$$

That is, we find the Fierz coefficient a^D_{IK} by summing the entries in the character table beneath a particular Γ^K where the rows form the elements of the particular Lorentz-covariant class $\Gamma^{(I)}$. This is readily seen by looking at table 2 for the $D=2$ example.

To complete the identification of (24) with (3) we use (22). For a specific K we can divide the D space–time indices into the disjoint sets: those appearing in K and the $D-K$ which are not. Then in choosing the indices on A we can take j from the K pile and $I-j$ from the other. Thus

$$a^D_{IK} = (-1)^{IK} \sum_{j=0}^{I} (-1)^j \binom{D-K}{I-j}\binom{D}{j}. \tag{25}$$

Then by the results of § 2 the lemma is estrablished.

4. Discussion

Thus far we have established the lemma (§ 2) and shown the connection with the characters of a finite group associated with the spin representation (§ 3). These approaches enable several insights.

Firstly, the orthogonality properties of the rows and columns of a Fierz matrix have always been reminiscent of the properties of a character table. Here we see why. That the Fierz matrix is Hermitian and unitary follows immediately from its definition in terms of characters.

A second connection with existing research is also enabled. The finite group here, as well as the orthogonal group, is simply reducible (Biedenharn 1981). That is, every element of G is equivalent to its inverse and the Kronecker product of two representations is multiplicity free. It has been hoped to find a geometric interpretation for the Clebsch–Gordan coefficients for such groups (Biedenharn 1981). In this vein a connection between the general $3n-j$ symbol and $\mathrm{PG}(n, 2)$, the projective geometry over the field of characteristic 2 (F_2) has been noted (Robinson 1970). In this regard we note that the one-dimensional characters of the finite groups described here form a vector space over F_2 and naturally associated with this we have a projective geometry. Actually the one-dimensional characters give us a Hadamard matrix which has an associated block design (Lander 1983). The appearance of these intrinsically interesting combinatoric structures suggests that a geometric interpretation may yet be found; the connections noted here are new and have yet to be fully explored.

It may be asked whether any of the periodic properties associated with spinors (Braden 1982) manifest themselves in the Fierz coefficients. The answer to this is no. The character tables of the different extra-special groups in our discussion are the same; the groups are isoclinic. This means that the Fierz coefficients derived from them are the same and so the periodicity properties are lost.

In conclusion then we have shown that the coefficients appearing in a Fierz transformation are characters of the symmetric group. Indeed, they are the sums of characters of the double cover of the reflection group upon which the symmetric group acts naturally—in fact just as it does on the Weyl group. This underlying character structure gives rise to the hermiticity properties of the transformation and enables connections to be made with an assortment of geometric structures.

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